

# $g = 2$ as a Gauge Condition

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## Abstract

Charged matter spin-1 fields enjoy a nonelectromagnetic gauge symmetry when interacting with vacuum electromagnetism, provided their gyromagnetic ratio is 2.

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It is agreed that the natural value for the gyromagnetic ratio  $g$  of an elementary charged particle coupling to the electromagnetic field  $F_{\mu\nu}$  is  $g = 2$  (in the absence of small radiative corrections), so that the Bargmann-Michel-Telegdi [1] equation of motion for the spin vector  $S_\mu$  takes its simplest form

$$\frac{dS_\mu}{d\tau} = \frac{e}{m} F_{\mu\nu} S^\nu \quad .$$

Further reasons for this choice can be given:

- (1) Within particle physics theories, Weinberg [2] has shown that the  $g = 2$  value must hold in tree approximation in order that scattering amplitudes possess good high-energy behavior. Furthermore, Ferrara, Porrati, and Telegdi [3] have implemented this requirement of good high-energy behavior in a Lagrangian framework, and regained  $g = 2$ .
- (2) Spin-1/2 charged leptons carry  $g = 2$ , and this confirms the view that they are “elementary” particles. The only known higher-spin, elementary charged particle – the W boson – possesses a gyromagnetic ratio consistent with  $g = 2$ . This is of course in agreement with the “standard model” where electromagnetism is fitted into a non-Abelian gauge group, which involves nonminimal electromagnetic coupling that results in  $g = 2$ . [4]

In this Comment there is offered yet another reason for preferring  $g = 2$  for charged vector mesons: The kinetic term in a manifestly Lorentz-invariant Lagrange density for spin-1 fields possesses a (nonelectromagnetic) gauge invariance that removes redundant field degrees of freedom, which do not propagate in the vacuum. This gauge invariance is preserved when the fields couple to external, *vacuum* electromagnetism ( $\partial^\mu F_{\mu\nu} = 0$ ) provided  $g = 2$ .

This simple observation is easily demonstrated. Consider the Lagrange density for free complex vector fields  $W_\mu$

$$\mathcal{L}_0 = -\frac{1}{2} |\partial_\mu W_\nu - \partial_\nu W_\mu|^2 \quad . \quad (1)$$

[In the following, we consider massless fields, or alternatively in the case of massive fields the discussion concerns the kinetic (derivative) portion of the Lagrange density.] Clearly  $\mathcal{L}_0$  possess the nonelectromagnetic gauge invariance

$$W_\mu \rightarrow W_\mu + \partial_\mu \xi \quad (2)$$

where  $\xi$  is complex.  $W_\mu$  is coupled to electromagnetism by replacing derivatives with covariant ones

$$\partial_\mu W_\nu \rightarrow D_\mu W_\nu \equiv (\partial_\mu + ieA_\mu)W_\nu \quad (3)$$

and allowing a further nonminimal interaction

$$\mathcal{L} = -\frac{1}{2} G^{*\mu\nu} G_{\mu\nu} + ie(g-1) F^{\mu\nu} W_\mu^* W_\nu \quad (4)$$

$$G_{\mu\nu} \equiv D_\mu W_\nu - D_\nu W_\mu \quad . \quad (5)$$

The nonminimal interaction ensures that charged vector particles carry gyromagnetic ratio  $g$ . When the electromagnetic field is source-free and  $g = 2$ , one verifies that  $\mathcal{L}$  remains invariant (up to total derivative terms) against the nonelectromagnetic gauge transformation (2), provided the derivatives are replaced by covariant ones,

$$W_\mu \rightarrow W_\mu + D_\mu \xi \quad . \quad (6)$$

As is well known, for massive fields with  $g = 2$  the transversality condition  $D^\mu W_\mu = 0$  follows from the Euler-Lagrange field equation, while in the massless case that condition can still be imposed thanks to the nonelectromagnetic gauge symmetry (6).

A similar situation holds for interactions with non-Abelian gauge potentials. When the vector meson fields form (in general) a complex multiplet  $W_\mu^i$ , which transforms under non-Abelian gauge transformations with the unitary representation matrices  $U^{ij}$

$$W_\mu^i \rightarrow (U^{-1})^{ij} W_\mu^j \quad (7)$$

whose anti-Hermitian generators are  $T_a$ ,  $[T_a, T_b] = f_{abc} T_c$ , then the gauge covariant derivative is

$$(D_\mu W_\nu)^i = \partial_\mu W_\nu^i + A_\mu^{ij} W_\nu^j \quad (8)$$

where the gauge potential is an element of the Lie algebra in this representation

$$A_\mu^{ij} = A_\mu^a T_a^{ij} \quad . \quad (9)$$

The Lagrange density for these fields reads (apart from a possible mass term)

$$\mathcal{L} = -\frac{1}{2}(G_{\mu\nu}^*)^i (G^{\mu\nu})^i + (g-1)(W^{\mu*})^i F_{\mu\nu}^{ij} (W^\nu)^j \quad . \quad (10)$$

A nonminimal coupling to the gauge field strength is present, and one verifies that the transformation

$$W_\mu \rightarrow W_\mu + D_\mu \Theta \quad (11)$$

changes  $\mathcal{L}$  only by total derivative terms when  $g = 2$  and the gauge fields are sourceless ( $D^\mu F_{\mu\nu} = 0$ ).

Additionally, let us note that in three-dimensional space-time and with vector fields in the adjoint representation, there exists another term that is invariant (apart from a total derivative) against (11). The relevant Lagrange density reads

$$\mathcal{L} = -\frac{1}{4}(G_{\mu\nu})^a (G^{\mu\nu})^a + f_{abc}(W^\mu)^a (W^\nu)^b F_{\mu\nu}^c + m\varepsilon^{\mu\alpha\beta} W_\mu^a F_{\alpha\beta}^a \quad . \quad (12)$$

Please observe that thanks to the Bianchi identity satisfied by  $F_{\alpha\beta}$  :  $D_\mu \varepsilon^{\mu\alpha\beta} F_{\alpha\beta} = 0$ , the last term possesses the symmetry (11) for arbitrary field strengths, not only sourceless ones. The

strength parameter  $m$  carries dimensions of mass and the term has been posited previously in a gauge- and parity-invariant mass generation mechanism for a gauge theory, where also the gauge transformation (11) was introduced [5].

The spin-2 case does not exhibit exactly the same behavior as above, yet something similar, but less direct, does hold. Without interactions, but with a mass term, the spin-2 equation of motion for a symmetric, second-rank tensor  $h_{\mu\nu}$  can be taken as

$$\begin{aligned} m^2(h_{\mu\nu} + \tfrac{1}{2}g_{\mu\nu}h) &= -\Box^2 h_{\mu\nu} + \partial_\mu h_\nu + \partial_\nu h_\mu - \partial_\mu \partial_\nu h \\ h_\nu &\equiv \partial^\mu h_{\mu\nu}, \quad h \equiv h^\mu_\mu \end{aligned} \quad (13)$$

and the right side (massless part) enjoys the nonelectromagnetic gauge invariance

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \theta_\nu + \partial_\nu \theta_\mu \quad . \quad (14)$$

Electromagnetic interactions can be included by promoting all derivatives to covariant ones, ordered in a specific fashion, and adding a nonminimal interaction. Thus one has

$$\begin{aligned} m^2(h_{\mu\nu} + \tfrac{1}{2}g_{\mu\nu}h) &= -D^2 h_{\mu\nu} + D_\mu h_\nu + D_\nu h_\mu \\ &\quad - \tfrac{1}{2}(D_\mu D_\nu + D_\nu D_\mu)h - ieg(F_{\mu\alpha}h_\nu^\alpha + F_{\nu\alpha}h_\mu^\alpha) \end{aligned} \quad (15)$$

where now  $h_\nu = D^\mu h_{\mu\nu}$ . With the ordering chosen above, the gyromagnetic ratio is  $g$ . Upon calculating the response of the right side to the gauge covariant version of the substitution (14)

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + D_\mu \theta_\nu + D_\nu \theta_\mu \quad (16)$$

one finds, with constant  $F_{\mu\nu}$  ( $\partial_\alpha F_{\mu\nu} = 0$ )

$$\Delta_{\mu\nu} = ie(2 - g)(F_{\mu\nu}D^\alpha \theta_\nu + F_{\nu\alpha}D^\alpha \theta_\mu) - ie(1 + g)(F_{\mu\nu}D_\nu \theta^\alpha + F_{\nu\alpha}D_\mu \theta^\alpha) \quad (17)$$

so that for *no* value  $g$  is invariance regained. [Note that for the “minimal” value [6]  $g = \frac{1}{2}$ ,

$$\Delta_{\mu\nu} = ie\frac{3}{2}(F_{\mu\alpha}(D^\alpha \theta_\nu - D_\nu \theta^\alpha) + F_{\nu\alpha}(D^\alpha \theta_\mu - D_\mu \theta^\alpha))$$

the nonvanishing response involves the antisymmetric combination  $D_\alpha \theta_\beta - D_\beta \theta_\alpha$ , while the definition of the transformation in (16) makes use of the symmetric combination  $D_\alpha \theta_\beta + D_\beta \theta_\alpha$ .] However, one may improve the situation by the following (rather artificial) consideration. Note that (16) implies that  $h_\mu$  transforms as

$$h_\nu \rightarrow D^2 \theta_\nu + D^\mu D_\nu \theta_\mu \quad (18)$$

that is, the ordering of the noncommuting covariant derivatives is inherited from previous definitions. But if we view  $h_\mu$  as an independent quantity, we can prescribe an arbitrary ordering in (18), which is equivalent to modifying (18) by a multiple of  $F_{\nu\mu}\theta^\mu$

$$\begin{aligned}
h_\nu &\rightarrow D^2\theta_\nu + D^\mu D_\nu\theta_\mu + iecF_{\nu\mu}\theta^\mu \\
&= D^2\theta_\nu + (1-c)D^\mu D_\nu\theta_\mu + cD_\nu D^\mu\theta_\mu \quad .
\end{aligned}
\tag{19}$$

Then the change in the kinetic part of the equation of motion (15) becomes

$$\Delta_{\mu\nu} = ie(2-g)(F_{\mu\alpha}D^\alpha\theta_\nu + F_{\nu\alpha}D^\alpha\theta_\mu) - ie(1+g-c)(F_{\mu\alpha}D_\nu\theta^\alpha + F_{\nu\alpha}D_\mu\theta^\alpha) \tag{20}$$

so that for the unique choice  $g = 2$  the nonelectromagnetic gauge invariance is maintained in the presence of constant fields, as long as the ordering on (19) is taken with  $c = 3$ . It remains an open question whether a more fundamental/natural reason can be found for this *ad hoc* ordering prescription.

## REFERENCES

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